Optimal Consumption Choice under Uncertainty with Intertemporal Substitution†

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Abstract

We extend the analysis of the intertemporal utility maximization problem for Hindy-Huang-Kreps utilities reported in Bank and Riedel (1998) to the stochastic case. Existence and uniqueness of optimal consumption plans are established under arbitrary convex portfolio constraints, including both complete and incomplete markets. For the complete market setting, Kuhn-Tucker-like necessary and sufficient conditions for optimality are given. Using this characterization, we show that optimal consumption plans are obtained by reflecting the associated level of satisfaction on a stochastic lower bound. When uncertainty is generated by a Lévy process and agents exhibit constant relative risk aversion, closed-form solutions are derived. Depending on the structure of the underlying stochastics, optimal consumption occurs at rates, in gulps, or singular to Lebesgue measure.

Keywords: Hindy-Huang-Kreps preferences, non-time additive utility optimization, intertemporal utility, intertemporal substitution

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Introduction

Standard intertemporal choice theory has long assumed that agents derive their satisfaction directly from the *rate of consumption*. Hindy, Huang, and Kreps (1992) (and, for the stochastic case, Hindy and Huang (1992)) strikingly show that such a specification of intertemporal preferences does not capture the substitutability of consumption over time, the reason being that the rate of consumption reacts too sensitively to small changes of the lifetime consumption plan. As a remedy Hindy, Huang, and Kreps (1992) proposed to replace the rate of consumption with some *level of satisfaction* which keeps track of past consumption. Indeed, preferences based on the latter quantity treat consumption of one and the same good at different but nearby times as substitutes.

Once this new approach to intertemporal choice theory has been accepted, it is important to understand the consumption behavior such preferences induce. This is the aim of the present paper, which extends our previous work, Bank and Riedel (1998), on Hindy–Huang–Kreps-preferences under certainty to the choice problem under uncertainty.

Approaching the utility maximization problem at first from a general perspective, we establish existence and uniqueness of optimal consumption plans. Existence of a solution to the utility maximization problem is an issue in the stochastic framework, since budget sets are no longer compact as in the deterministic setting. Using a new method which is based on a theorem of Komlós (1967) and its extension by Kabanov (1999), we are able to give a short proof of existence of an optimal policy under convex portfolio constraints. This includes the cases of complete as well as incomplete markets, and extends the result of Jin and Deng (1997) who prove existence for the special case of short-sale constraints in a diffusion model.

Moving on, we study the characterization and construction of optimal consumption plans when markets are complete. In this context, Hindy and Huang (1993) use the *Bellmann methodology* to derive sufficient conditions for optimality based on the Hamilton-Jacobi-Bellmann equation; in a special case, this allows them to give an explicit solution. Instead of using the Bellman approach, we extend the *Kuhn-Tucker-like theorem* of Bank and Riedel (1998) from the deterministic to the stochastic framework, thereby obtaining necessary and sufficient conditions for optimality. This way of attacking the problem is similar to the Cox and Huang (1989) method in the time-additive case.

The explicit construction of optimal plans is more difficult than in the deterministic case, where the optimal level of satisfaction is a smooth time-dependent function of the current price for consumption. Since the present context allows for price processes of unbounded variation, the optimal level of satisfaction can no longer be of this form, because — as an average — it typically has bounded variation.

Proceeding from our characterization of optimal plans, we derive an equation (cf. (17)) characterizing what we call the *minimal level of satisfaction*, and show that the investor optimally consumes just enough to keep his level of satisfaction always above this minimum. This allows us to reduce the utility optimization problem to finding a solution to the *minimal level equation*, which, therefore, plays here the same role as the Hamilton-Jacobi-Bellman equation in the dynamic programming approach.

In a homogeneous setting where in particular prices are driven by a Lévy process, we provide an explicit solution to the minimal level equation. This gives the explicit description of the optimal consumption plan and allows us to calculate the indirect utility in closed-form.

We carry out several case studies illustrating the considerable flexibility of the Hindy-Huang-Kreps framework. The whole variety of consumption patterns can arise, depending only on the structure of the respectively chosen stochastics. If state prices are driven by Brownian motion, optimal consumption is singular with respect to Lebesgue measure, as already pointed out by Hindy and Huang (1993). If prices are driven by a Poisson process, the occurring price shocks or — in the terminology of Hindy and Huang (1992) — the corresponding information surprises induce the investor to consume in gulps, if there is a 'nice' downward price shock. If prices jump upward, he refrains from consumption for a while, until he has got 'unsatisfied' or rich enough to make him willing to consume again.

An outline of the paper is as follows. In Section 1 we introduce the general technical framework and formulate the utility maximization problem. Section 2 proves existence and uniqueness of a solution. In Section 3 we give necessary and sufficient conditions for optimality when the considered financial market is complete. Furthermore, we investigate the general structure of optimal consumption plans and motivate the concept of the 'minimal level of satisfaction'. Finally, Section 4 provides some explicit solutions.

1 Formulation of the Utility Maximization Problem

Consider an investor who wishes to consume his initial wealth $w \geq 0$ over a fixed finite time period $[0, T]$. Assume he can invest in a money market account with interest rate $r = (r(t),\ 0 \leq t \leq T)$, a bounded adapted process, and in at least one risky security. Uncertainty is described by a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t,\ 0 \leq t \leq T), \mathbb{P})$ satisfying the usual conditions of right continuity and completeness; $\mathcal{F}_0$ is $\mathbb{P}$-a.s. trivial. A priori, the consumption plans at the investor's deposit are given by

$$\mathcal{X}_+ \stackrel{\Delta}{=} \{C \mid C \text{ is the distribution function of an optional random measure}^1\},$$

while his budget-feasible set is

$$\mathcal{A}(w) \stackrel{\Delta}{=} \{C \in \mathcal{X}_+ \mid \Psi(C) \leq w\}\,.$$

[1]Equivalently, $C : \Omega \times [0, T] \to \mathbb{R}$ is a nonnegative, adapted process with increasing, rightcontinuous paths.

Here, $\Psi(C) \in [0, \infty]$ denotes the minimal initial capital needed to finance a given consumption plan $C \in \mathcal{X}_+$ by investing in the assets of the financial market. We assume this quantity can be expressed in the form [2]

$$\Psi(C) \stackrel{\Delta}{=} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^* \int_0^T \gamma(t)\, dC(t) \quad (C \in \mathcal{X}_+) \tag{1}$$

where $\gamma(t) \stackrel{\Delta}{=} \exp\left(-\int_0^t r(s)\, ds\right)$ and $\mathcal{P}$ is a fixed nonempty set of probability measures on $(\Omega, \mathcal{F}_T)$. The specific choice of this set is determined by the risk-structure of the considered financial market. The elements of $\mathcal{P}$ are called risk-neutral measures.

Remark 1.1 *Note that the above formulation allows for incomplete markets and, more generally, even for markets under convex constraints; see, e.g., Föllmer and Kabanov (1998), Föllmer and Kramkov (1997), Cvitanic and Karatzas (1993).*

To illustrate this further, let us consider a model of a security market consisting of a riskless bond and a stock. Assume short selling of the stock is prohibited. Föllmer and Kramkov (1997) show that this economic setting may be captured by choosing

$$\mathcal{P} \stackrel{\Delta}{=} \{\mathbb{P}^* \sim \mathbb{P} \mid \mathbb{P}^* \text{ is a supermartingale measure for each } S \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the set of all gain processes which are attainable by some admissible strategy without short selling. More precisely, they prove that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\gamma(T)H]$$

is the minimal amount needed to hedge a given contingent claim $H \geq 0$ with maturity T. For a consumption plan $C \in \mathcal{X}_+$, this induces formula (1) *for the minimal budget the investor needs to finance it.*

With a given consumption plan $C \in \mathcal{X}_+$ the investor associates the utility

$$U(C) \stackrel{\Delta}{=} \int_0^T u(t, Y(C)(t))\, dt$$

where $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ denotes a continuous felicity function which is increasing and concave in its second argument, and where

$$Y(C)(t) \stackrel{\Delta}{=} \gamma(t) + \int_0^t \theta(t, s)\, dC(s)$$

[2] By convention, integration over time intervals is carried out including the involved finite boundaries. We let any consumption stream start in $C(0-) \stackrel{\Delta}{=} 0$; a positive value at time 0 indicates an initial consumption gulp and corresponds to a point mass $C(0) > 0$ of the random measure dC at time $t = 0$. Similarly, we henceforth assume that any other integrator B starts from some initial value $B(0-)$ assumed to be zero unless otherwise stated.

is the investor's level of satisfaction obtained from his consumption up to time $t \in [0,T]$. We assume that the deterministic functions $y : [0,T] \to \mathbb{R}$ and $\theta : [0,T] \times [0,T] \to \mathbb{R}$ are continuous and nonnegative. $\theta(t,s)$ describes the weight assigned at time t to consumption made at time $s \le t$; $y(t)$ may be interpreted as an exogenously given level of satisfaction for time t.

Remark 1.2 *A standard choice for θ and y is $\theta(t,s) \triangleq \beta e^{-\beta(t-s)}$ and $y(t) \triangleq \eta e^{-\beta t}$ with constants $\beta, \eta > 0$; compare, e.g., Sundaresan (1989), Constantinides (1990).*

The investor's problem is to maximize his expected utility over all budget-feasible consumption plans, i.e.,

$$\max_{C \in \mathcal{A}(w)} \mathbb{E} U(C). \tag{2}$$

2 Existence and Uniqueness

This section is devoted to the proof of existence and uniqueness of a solution for the utility maximization problem (2) under

Assumption 2.1 *We have*

(i) the felicity function u is bounded,

or

(ii) the felicity function satisfies the growth-condition

$$u(t,y) \le A\left(1 + y^{\alpha}\right) \quad \forall\ 0 \le t \le T, y \ge 0 \tag{3}$$

for some constants $A > 0, \alpha \in (0,1)$ and there is a risk-neutral measure $\hat{\mathbb{P}} \in \mathcal{P}$ with density $\hat{Z} \triangleq \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ satisfying

$$\hat{Z}^{-1} \in \mathcal{L}^{\hat{p}}(\mathbb{P}) \tag{4}$$

for some $\hat{p} > \frac{\alpha}{1-\alpha}$.

Remark 2.2 *Similar assumptions on felicity functions have been made for the case of time-additive functionals in Cox and Huang (1991) and Aumann and Perles (1965). The example in Kramkov and Schachermayer (1998) suggests that it is in general impossible to weaken our assumption. An integrability condition similar to (4) can be found in Cuoco (1997).*

The following is the main result of this section.

Theorem 2.3 *Under Assumption 2.1, the utility maximization problem* (2) *has a solution. This solution is unique if, in addition, $u(t,.)$ is strictly concave for every $t \in [0,T]$ and $C \mapsto Y(C)$ is injective up to $\mathbb{P}$-indistinguishability.*

Remark 2.4 *Injectivity of $C \mapsto Y(C)$ follows, e.g., if $\theta(t,s) = \kappa_1(t)\kappa_2(s)$ for some strictly positive, continuous functions $\kappa_1, \kappa_2 : [0,T] \to \mathbb{R}$.*

Let us prepare the proof of Theorem 2.3 by the following technical

Lemma 2.5 *(i) There is a constant $B > 0$ such that*

$$Y(C)(t) \leq B(1 + C(t)) \quad (0 \leq t \leq T)$$

for all $C \in \mathcal{X}_+$.

(ii) If $C^n \in \mathcal{X}_+$ $(n = 1, 2, \dots)$ converge weakly to $C \in \mathcal{X}_+$ then

$$Y(C^n)(t) \to Y(C)(t)$$

for every point of continuity t of C.

(iii) $\mathcal{A}(w)$ is norm-bounded uniformly in $L^1(\mathbb{P}^*)$ $(\mathbb{P}^* \in \mathcal{P})$, i.e.,

$$\sup_{\mathbb{P}^* \in \mathcal{P}} \sup_{C \in \mathcal{A}(w)} \mathbb{E}^* C(T) < +\infty.$$

(iv) $\{U(C), C \in \mathcal{A}(w)\}$ is uniformly $\mathbb{P}$-integrable.

Proof : (i) and (ii) follow immediately from our assumptions on $y(.)$ and $\theta(.,.)$. The boundedness of the interest rate process r implies (iii). It remains to prove (iv). This assertion is trivial if u is bounded. For unbounded u we show that $\{U(C), C \in \mathcal{A}(w)\}$ is bounded in $L^p(\mathbb{P})$ with $p \triangleq \frac{\hat{p}}{\alpha(1+\hat{p})} > 1$. Indeed, in this case Assumption 2.1 and (i) yield

$$\mathbb{E}\left[U(C)^p\right] \leq \mathbb{E}\left(\int_0^T A(1 + Y(C)(t)^\alpha)\, dt\right)^p \leq c\,(1 + \mathbb{E}\left[C(T)^{\alpha p}\right])$$

for some constant $c > 0$. Note that $\alpha p < 1$, and apply Hölder's inequality to get

$$\mathbb{E}\left[C(T)^{\alpha p}\right] \leq \mathbb{E}\left[C(T)\hat{Z}\right]^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\frac{\alpha p}{1-\alpha p}}\right]^{1-\alpha p} \leq c w^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\hat{p}}\right]^{1-\alpha p},$$

for some constant $c > 0$ by (iii). In connection with the above estimation and (4), this yields the desired $L^p(\mathbb{P})$-boundedness. □

Remark 2.6 *Since uniform integrability implies $\mathcal{L}^1$-boundedness, part (iv) of the above Lemma yields in particular that the value of our maximization problem* (2) *is finite.*

Now we can give the

Proof of Theorem 2.3 Choose a maximizing sequence $\tilde{C}^n \in \mathcal{A}(w)$ $(n = 1, 2, \dots)$ for (2). By Lemma 2.5 (iii) and Kabanov's version of Komlós' Theorem (Kabanov (1999), Lemma 3.5; Komlós (1967)), there exists a subsequence, again denoted by $\left(\tilde{C}^n\right)$, which is almost surely weakly Cesaro convergent to some $C^* \in \mathcal{X}_+$, i.e., almost surely we have

$$C^n(t) \stackrel{\Delta}{=} \frac{1}{n} \sum_{k=1}^{n} \tilde{C}^k(t) \to C^*(t) \quad (n \uparrow +\infty)$$

for every point of continuity t of C^*. By concavity, (C^n) is again a maximizing sequence. We claim that C^* is optimal for (2). Indeed, since γ is continuous, we have

$$\int_0^T \gamma(t)\, dC^*(t) = \lim_n \int_0^T \gamma(t)\, dC^n(t) \quad \mathbb{P}\text{-a.s.}$$

Hence, by Fatou's Lemma,

$$\mathbb{E}^* \int_0^T \gamma(t)\, dC^*(t) \leq \liminf_n \mathbb{E}^* \int_0^T \gamma(t)\, dC^n(t) \leq w,$$

for every $\mathbb{P}^* \in \mathcal{P}$, i.e., $C^* \in \mathcal{A}(w)$. Furthermore, Lemma 2.5 (i) and (ii) yield $U(C^n) \to U(C^*)$ for $n \uparrow +\infty$ $\mathbb{P}$-a.s. by dominated convergence. In connection with part (iv) of the same lemma, this implies optimality of C^* by Lebesgue's Theorem.

If two solutions $\tilde{C}$ and C^* are not indistinguishable, then, by assumption, neither are their respective levels of satisfaction $\tilde{Y} \stackrel{\Delta}{=} Y(\tilde{C})$ and $Y^* \stackrel{\Delta}{=} Y(C^*)$. Optimality excludes that these levels only differ at time $t = T$ because this would imply a (suboptimal) final jump by one of the policies. Thus, on a set with positive probability $\tilde{Y}$ and Y^* differ on an open time interval. Hence, by strict concavity of $u(t, .)$ for any $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}U\left(\tfrac{1}{2}\{\tilde{C} + C^*\}\right) &= \mathbb{E}\int_0^T u\left(t, \tfrac{1}{2}\{\tilde{Y}(t) + Y^*(t)\}\right) dt \\
&> \mathbb{E}\int_0^T \tfrac{1}{2}\left\{u(t, \tilde{Y}(t)) + u(t, Y^*(t))\right\} dt \\
&= \tfrac{1}{2}\{\mathbb{E}U(\tilde{C}) + \mathbb{E}U(C^*)\} \\
&= \max_{C \in \mathcal{A}(w)} \mathbb{E}U(C)
\end{aligned}$$

in contradiction to $\frac{1}{2}\{\tilde{C} + C^*\} \in \mathcal{A}(w)$ and to the optimality of $\tilde{C}$ and C^* in this set. □

3 Solutions in the Complete Case

Assume now that the financial market is complete in the sense that $\mathcal{P} = \{\hat{\mathbb{P}}\}$ is a singleton. Thus there is precisely one risk neutral measure $\mathbb{P}^* = \hat{\mathbb{P}}$. Set

$\hat{Z}(t) \triangleq \mathbb{E}\left[\hat{Z} \,\middle|\, \mathcal{F}_t\right]$ $(0 \le t \le T)$ and suppose that $\hat{Z}$ satisfies the integrability condition (4). Let $\psi(t) \triangleq y(t)\hat{Z}(t)$ $(0 \le t \le T)$ denote the associated state-price density. From now on, we require, in addition to Assumption 2.1, the following

Assumption 3.1 *The felicity function $u = u(t, y)$ is strictly concave and differentiable in y.*

Since a strictly concave and increasing function is strictly increasing, the above assumption ensures that the investor's utility function is non-satiated. Hence, he will always exhaust his budget.

3.1 Necessary and Sufficient Conditions for Optimality

In the complete setting described above, we can formulate and prove the following analog of the Kuhn-Tucker Theorem for the utility maximization problem (2). It provides a characterization of the solution to (2) in terms of the functional

$$\phi(C)(t) \triangleq \mathbb{E}\left[\int_t^T \partial_y u(s, Y(C)(s))\theta(s,t)\, ds \,\middle|\, \mathcal{F}_t\right] \quad (0 \le t \le T, C \in \mathcal{X}_+). \tag{5}$$

Remark 3.2 *$\phi(C)(t)$ may be interpreted as marginal utility resulting from an additional infinitesimal consumption at time t, otherwise following the consumption plan $C \in \mathcal{X}_+$. Note furthermore that $\phi(C)(t)$ is always well-defined as a random variable taking values in $[0, +\infty]$ because $\partial_y u$ and θ are nonnegative. Mathematically, $\phi(C)$ may be interpreted as the Riesz representation of the utility gradient at C, as pointed out by Duffie and Skiadas (1994) in their Example 5.*

Theorem 3.3 *A consumption plan $C^* \in \mathcal{X}_+$ solves* (2) *if and only if the following conditions hold true for some Lagrange multiplier $M \ge 0$:*

(i) $\Psi(C^*) = w$,

(ii) $\phi(C^*) \le M\psi$ $\mathbb{P}$*-a.s.,*

(iii) C^* *is flat off* $\{\phi(C^*) = M\psi\}$ $\mathbb{P}$*-a.s., i.e.,*

$$\mathbb{E}\int_0^T 1_{\{\phi(C^*)(t) \ne M\psi(t)\}}\, dC^*(t) = 0\,.$$

PROOF : Let us first prove sufficiency. Take a budget-feasible $C \in \mathcal{A}(w)$ and let $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$. By concavity of u and by definition of Y and Y^*, one has

$$\begin{aligned}
\mathbb{E}U(C^*) - \mathbb{E}U(C) &= \mathbb{E}\int_0^T \{u(s, Y^*(s)) - u(s, Y(s))\}\, ds \\
&\ge \mathbb{E}\int_0^T \{\partial_y u(s, Y^*(s))(Y^*(s) - Y(s))\}\, ds \\
&= \mathbb{E}\int_0^T \left\{\partial_y u(s, Y^*(s)) \int_0^s \theta(s,t)\,[dC^*(t) - dC(t)]\right\} ds\,.
\end{aligned}$$

We split up the last expectation in two terms:

$$I^* \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \theta(s,t)\, dC^*(t) \right\} ds$$

and

$$I \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \theta(s,t)\, dC(t) \right\} ds\,.$$

For the first term Fubini's Theorem yields

$$I = \mathbb{E} \int_0^T \left\{ \int_t^T \partial_y u(s, Y^*(s)) \theta(s,t)\, ds \right\} dC(t)\,.$$

Since C is adapted, we may replace the $\{\dots\}$-term in the above expression by its conditional expectation which, by definition, is $\phi(C^*)$; this follows, e.g., from Lemma I.3.12 in Jacod and Shiryaev (1987). Hence,

$$I = \mathbb{E} \int_0^T \phi(C^*)(t)\, dC(t) \le M \mathbb{E} \int_0^T \psi(t)\, dC(t) \le Mw$$

where the first inequality follows from condition (ii) and the last inequality is due to the budget constraint. By conditions (i) and (iii), the above calculation carried out for C^* instead of C shows

$$I^* = \mathbb{E} \int_0^T \phi(C^*)(t)\, dC^*(t) = M \mathbb{E} \int_0^T \psi(t)\, dC^*(t) = Mw.$$

Summing up, we find

$$\mathbb{E}U(C^*) - \mathbb{E}U(C) \ge I^* - I \ge Mw - Mw = 0$$

establishing sufficiency. Necessity follows from Lemma 3.4 and Lemma 3.5 below. □

Before we attack the necessity part of the proof of Theorem 3.3, let us briefly sketch the argument. The idea is to proceed along the same lines as in the proof of the finite-dimensional Kuhn-Tucker Theorem. In a first step, we show that the candidate policy C^* solves the problem linearized around C^*. This is done in Lemma 3.4 below. Solutions of the linear problem are easily characterized (Lemma 3.5), and it follows that C^* has to satisfy the conditions given in Theorem 3.3.

Lemma 3.4 *Let $C^* \in \mathcal{A}(w)$ be optimal for* (2) *and let $\phi^* \triangleq \phi(C^*)$. Then C^* solves the linear problem*

(6) $$\max_{C \in \mathcal{A}(w)} \mathbb{E} \int_0^T \phi^*(t)\, dC(t)\,,$$

and the value of this problem is finite.

PROOF : Consider $C \in \mathcal{A}(w)$ and let $C^\epsilon \triangleq \epsilon C + (1-\epsilon)C^*$ $(0 \le \epsilon \le 1)$. By optimality of C^* and concavity of $u(t,.)$ $(0 \le t \le T)$, we have for $Y^\epsilon \triangleq Y(C^\epsilon)$, $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$

$$\begin{aligned}
0 &\ge \frac{1}{\epsilon}\{\mathbb{E}U(C^\epsilon) - \mathbb{E}U(C^*)\} \\
&= \mathbb{E}\int_0^T \frac{1}{\epsilon}\{u(s,Y^*(s)+\epsilon(Y(s)-Y^*(s))) - u(s,Y^*(s))\}ds \\
&\ge \mathbb{E}\int_0^T \{\partial_y u(s,Y^\epsilon(s))(Y(s)-Y^*(s))\}\,ds \\
&= \mathbb{E}\int_0^T \{\partial_y u(s,Y^\epsilon(s))\int_0^s \theta(s,t)\,[dC(t)-dC^*(t)]\}\,ds \\
&= \mathbb{E}\int_0^T \{\int_t^T \partial_y u(s,Y^\epsilon(s))\theta(s,t)\,ds\}\,[dC(t)-dC^*(t)] \\
&= \mathbb{E}\int_0^T \Phi^\epsilon(t)\,[dC(t)-dC^*(t)],
\end{aligned}$$

where $\Phi^\epsilon(t) \triangleq \int_t^T \partial_y u(s,Y^\epsilon(s))\theta(s,t)\,ds$ $(0 \le t \le T)$. Let furthermore $\Phi^* \triangleq \Phi^0$. By Fatou's Lemma we have

$$\liminf_{\epsilon\downarrow 0}\mathbb{E}\int_0^T \Phi^\epsilon(t)\,dC(t) \ge \mathbb{E}\int_0^T \Phi^*(t)\,dC(t). \tag{7}$$

We claim that

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\int_0^T \Phi^\epsilon(t)\,dC^*(t) = \mathbb{E}\int_0^T \Phi^*(t)\,dC^*(t) \tag{8}$$

whence we may deduce our assertion as follows. Letting $\epsilon \downarrow 0$ in the above series of estimations, (7) and (8) give us

$$\mathbb{E}\int_0^T \Phi^*(t)\,dC(t) \le \mathbb{E}\int_0^T \Phi^*(t)\,dC^*(t).$$

Since C and C^* are adapted, Lemma I.3.12 of Jacod and Shiryaev (1987) allows us to replace the term $\Phi^*(t)$ in the above inequality by its conditional expectation $\phi^*(t)$, and we obtain indeed optimality of C^* for the linear problem (6).

It remains to prove (8). For this it suffices to show that

$$I^\epsilon \triangleq \int_0^T \Phi^\epsilon(t)\,dC^*(t) \quad (0 \le \epsilon \le \tfrac{1}{2})$$

has a $\mathbb{P}$-integrable upper bound. In the following chain of inequalities, we use the estimate $\partial_y u(t,y)y \le u(t,y) - u(t,0)$ for the concave felicity function and the estimates $Y^\epsilon(s) \geq \frac{1}{2}Y^*(s)$, $\int_0^s \theta(s,t)\,dC^*(t) \leq Y^*(s)$ for the level of satisfaction:

$$\begin{aligned}
I^\epsilon &= \int_0^T \left\{\partial_y u(s, Y^\epsilon(s)) \int_0^s \theta(s,t)\,dC^*(t)\right\} ds \\
&\leq \int_0^T \partial_y u(s, Y^\epsilon(s)) Y^*(s)\,ds \\
&\leq 2\int_0^T \partial_y u(s, \tfrac{1}{2}Y^*(s))\tfrac{1}{2}Y^*(s)\,ds \\
&\leq 2\int_0^T \left(u(s, \tfrac{1}{2}Y^*(s)) - u(s,0)\right) ds \\
&\leq 2\int_0^T \left(u(s, Y^*(s)) - u(s,0)\right) ds \\
&= 2\left\{U(C^*) - \int_0^T u(s,0)\,ds\right\}.
\end{aligned}$$

Since $U(C^*)$ is integrable by Lemma 2.5 (iv), we have found the required upper bound for I^ϵ. □

Let us now discuss the linear problem (6).

Lemma 3.5 *Let ϕ, ψ be two strictly positive, rightcontinuous and adapted processes. Then every solution C^* to the linear optimization problem*

$$\max_{C \in X_+} \mathbb{E}\int_0^T \phi(t)\,dC(t) \quad s.t. \quad \mathbb{E}\int_0^T \psi(t)\,dC(t) \leq w \tag{9}$$

satisfies

$$\mathbb{E}\int_0^T 1_{\{\phi(t) \neq M\psi(t)\}}\,dC^*(t) = 0, \tag{10}$$

where

$$M \triangleq \operatorname{ess\,sup}_{\Omega} \sup_{t\in[0,T]} \frac{\phi(t)}{\psi(t)}.$$

Proof :

1. We first show that the value v of the linear problem (9) is given by Mw. Indeed, it is easy to see that $v \leq Mw$. On the other hand, for every $K < M$ the set

$$\left\{\omega \in \Omega \;\middle|\; \left(\sup_{t\in[0,T]} \frac{\phi(t)}{\psi(t)}\right)(\omega) > K\right\}$$

has positive probability. Therefore, letting

$$\tau^K \triangleq \inf\left\{t \in [0,T] \;\middle|\; \frac{\phi(t)}{\psi(t)} > K\right\}$$

we can find $c \geq 0$ such that $C^K \triangleq c1_{[\tau^K,T]} \in \mathcal{X}_+$ satisfies $\mathbb{E}\int_0^T \psi(t)\,dC^K(t) = w$. We have

$$\begin{aligned} Mw \;\geq\; v \;&\geq\; \mathbb{E}\int_0^T \phi(t)\,dC^K(t) = \mathbb{E}\left[c\phi(\tau^K)1_{\{\tau^K<+\infty\}}\right] \\ &\geq\; \mathbb{E}\left[cK\psi(\tau^K)1_{\{\tau^K<+\infty\}}\right] = K\mathbb{E}\int_0^T \psi(t)\,dC^K(t) \\ &=\; Kw. \end{aligned}$$

Letting $K \uparrow M$ in the above inequality yields $v = Mw$.

2. Suppose that C^* is a solution to (9). Then by 1. and the definition of M

$$Mw = \mathbb{E}\int_0^T \phi(t)\,dC^*(t) \leq M\mathbb{E}\int_0^T \psi(t)\,dC^*(t) \leq Mw$$

implying (10).

□

3.2 The Structure of Optimal Consumption Plans

As the finite-dimensional Kuhn-Tucker Theorem, our infinite-dimensional version (Theorem 3.3) does not yield an explicit description of the optimum. However, we can use the obtained characterization to analyze the general structure of the solution, as we will show in this section. The main result of this analysis will be Theorem 3.12. This theorem provides an equation characterizing what we call the 'minimal level of satisfaction'. This is an adapted process $I = (I(t),\ 0 \leq t \leq T)$ which gives us a canonical lower bound at which the investor should optimally 'reflect' his level of satisfaction. In our non-Markovian setup, the equation characterizing this level plays the same role as the Hamilton-Jacobi-Bellmann equation does in Dynamic Programming.

As a first application of Theorem 3.3, we prove a version of the Dynamic Programming Principle:

Proposition 3.6 *If $C^* \in \mathcal{X}_+$ is a solution to* (2) *then,* $\mathbb{P}$*-a.s., it also solves the problem*

$$\text{Maximize } \mathbb{E}\left[\,U(C)\,|\,\mathcal{F}_t\right] \text{ subject to } C \equiv C^* \text{ on } [0,t) \text{ and } \Psi_t(C) \leq \Psi_t(C^*)$$

where

$$\Psi_t(C) \triangleq \frac{1}{\gamma(t)}\mathbb{E}\left[\left.\int_t^T \gamma(s)\,dC(s)\,\right|\,\mathcal{F}_t\right] \quad (C \in \mathcal{X}_+)$$

is the price-functional at time t. Thus, a consumption plan which is optimal at time zero is its best continuation at any other time $t > 0$.

PROOF : Using the first-order conditions satisfied by C^*, this can be shown by the same calculation as for the sufficiency part of Theorem 3.3, now carried out for conditional expectations instead of ordinary expectations. $\square$

Let us now study the dependency of the optimal consumption plan on the exogenous level of satisfaction $y(.)$. To make this precise, let us specify the following dynamics for the level of satisfaction:

Assumption 3.7 *For a consumption plan $C \in \mathcal{X}_+$, the corresponding level of satisfaction $Y(C)$ evolves according to the ODE*

$$(11) \quad Y(C)(0-) = \eta, \quad dY(C)(t) = \beta(t)\,(dC(t) - Y(C)(t-)\,dt) \quad (0 \le t \le T)$$

where $\beta(.)$ is a strictly positive, continuous function $[0,T] \to \mathbb{R}$ and $\eta \ge 0$ is a constant.

Remark 3.8 *The (unique) solution to* (11) *is given by*

$$Y(C)(t) = \eta e^{-\int_0^t \beta(s)\,ds} + \int_0^t \beta(s) e^{-\int_s^t \beta(v)\,dv}\, dC(s) \quad (0 \le t \le T)\,.$$

Thus, the choice

$$y(t) \triangleq \eta e^{-\int_0^t \beta(s)\,ds}, \quad \theta(t,s) \triangleq \beta(s) e^{-\int_s^t \beta(v)\,dv} \quad (0 \le s \le t \le T)$$

imbeds the above specification of $Y(.)$ into the context of the preceding sections.

Under Assumption 3.7, $C \mapsto Y(C)$ is injective and, therefore, we may apply Theorem 2.3 to obtain existence and uniqueness of an optimal consumption plan for every choice of the initial level of satisfaction $\eta \ge 0$. In order to stress its dependency on this parameter, let us denote this plan by $C^{M,\eta}$; $M > 0$ is the Lagrange multiplier induced by our Kuhn-Tucker Theorem 3.3.

Now, we may ask: How does the optimal plan $C^{M,\eta}$ depend on the initial level of satisfaction η?

Lemma 3.9 *Let $Y(.)$ and $\tilde{Y}(.)$ denote the functionals for the level of satisfaction with initial value η and $\tilde{\eta}$, respectively. Suppose* $0 \le \eta \le \tilde{\eta}$.

Then the respective optimal levels of satisfaction $Y^* \triangleq Y(C^{M,\eta})$, $\tilde{Y}^* \triangleq \tilde{Y}(C^{M,\tilde{\eta}})$ with the same *Lagrange multiplier $M > 0$ are related by*

$$(12) \qquad \tilde{Y}^*(t) = \tilde{\eta} e^{-\int_0^t \beta(s)\,ds} \vee Y^*(t) \quad (0 \le t \le T)\,.$$

In particular, we have

$$(13) \qquad dC^{M,\tilde{\eta}}(t) = 1_{\{\tau < t \le T\}}\, dC^{M,\eta}(t) + \tilde{\Delta}\,\delta_{\{\tau\}}(dt)$$

where the second summand is the Dirac measure with point mass

$$\tilde{\Delta} \triangleq Y^*(\tau) - \tilde{\eta} e^{-\int_0^\tau \beta(s)\,ds}$$

at time

$$\tau \triangleq \inf \left\{ t \ge 0 \;\middle|\; \tilde{\eta} e^{-\int_0^t \beta(s)\,ds} \ge Y^*(t) \right\}.$$

PROOF : Let $\tilde{C} \in \mathcal{X}_+$ be the consumption plan defined by the right side of (13). From the dynamics for the level of satisfaction specified in Assumption 3.7, it may easily deduced that $\tilde{Y}(\tilde{C})$ coincides with the right side of (12). Moreover, we see that $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, T]$. We will show that $\tilde{C}$ is optimal for the problem with initial level of satisfaction $\tilde{\eta}$ and that it has Lagrange multiplier $M > 0$. By uniqueness of this plan, we then obtain Equation (12).

So let us verify the first-order conditions for $\tilde{C}$. For $0 \leq t \leq T$ we have

$$\begin{aligned}
\tilde{\phi}(\tilde{C})(t) &= \mathbb{E}\left[\int_t^T \partial_y u\left(s, \tilde{Y}(\tilde{C})(s)\right)\theta(s,t)\,ds \,\middle|\, \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\int_t^T \partial_y u\left(s, Y^*(s)\right)\theta(s,t)\,ds \,\middle|\, \mathcal{F}_t\right] = \phi(C^{M,\eta})(t) && (14)\\
&\leq M\psi(t)\,, && (15)
\end{aligned}$$

where Inequality (14) follows from $\tilde{Y}(\tilde{C}) \geq Y^*$; Inequality (15) is due to the first-order conditions satisfied by $C^{M,\eta}$. The above estimate shows that $\tilde{C}$ satisfies the first-order inequality constraint with Lagrange multiplier $M > 0$.

Hence, it remains to check the flat-off condition. Note first that supp $d\tilde{C} \subset [\tau, T]$. Moreover, we have $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, T]$ and, therefore, also $\tilde{\phi}(\tilde{C}) = \phi(C^{M,\eta})$ on this interval. Hence,

$$\mathbb{E}\int_0^T 1_{\{\tilde{\phi}(\tilde{C}) \neq M\psi\}}\,d\tilde{C} = \mathbb{E}\int_\tau^T 1_{\{\phi(C^{M,\eta}) \neq M\psi\}}\,d\tilde{C} = 0$$

where the last inequality is due to the absolute continuity of $d\tilde{C}$ with respect to $dC^{M,\eta}$ and to the flat-off condition satisfied by the latter consumption plan. □

Remark 3.10 *The preceding lemma shows in particular that it suffices to find the optimal consumption plan for $\eta = 0$. All other cases may be recovered from this one by Equations* (12) *and* (13).

Let us now motivate the announced concept of a 'minimal level of satisfaction' by some heuristics.

For every time $t \in [0, T)$, consider an agent, called t–Adam, who is born at time t. t–Adam starts with an initial level of satisfaction of zero. Taking the history $\mathcal{F}_t$ as given, he solves

$$\text{Maximize } \mathbb{E}\left[\int_t^T u\left(v, Y_t(C)(v)\right)\,dv \,\middle|\, \mathcal{F}_t\right] \text{ subject to } \Psi_t(C) \leq w_t^M\,,$$

where

$$Y_t(C)(v) \stackrel{\Delta}{=} \int_t^v \beta e^{-\beta(v-u)}\,dC(u) \quad (t \leq v \leq T)$$

denotes the evolution of t–Adam's level of satisfaction if, from his birth on, he follows the consumption plan C. We assume that, at his time of birth, t–Adam is endowed with the initial capital $w_t = w_t^M$ needed to buy the optimal consumption plan C_t^M which has Lagrange multiplier $M > 0$. This Lagrange multiplier is also shared by all his brothers.

Now imagine that s–Adam, with $s < t$, thinks about his consumption from time t on. We claim that he can deduce his optimal behavior from observing his younger brother t–Adam. In fact, as long as s–Adam's level of satisfaction $Y_s(.) \stackrel{\Delta}{=} Y_s(C_s^M)(.)$ is strictly higher than t–Adam's, he should not consume. Once s–Adam's level of satisfaction has dropped to t–Adam's level, it is optimal to mimic t–Adam's behavior[3].

Heuristically, we argue therefore that

$$I(t) \stackrel{\Delta}{=} Y_t(t) \quad (0 \le t < T)$$

is a *universal* lower bound from which we may recover *all* optimal consumption plans C_t^M with the same Lagrange multiplier $M > 0$ via 'reflection' of the level of satisfaction. Lemma 3.11 below makes precise what we mean by 'reflecting the level of satisfaction at a given lower bound'. We state this result only for initial time t being equal to zero, the general case $t \ge 0$ can be treated analogous. Figure 1 in Section 4.1 below illustrates the way a consumption plan may be defined by this property.

Lemma 3.11 *Let $I = (I(t), 0 \le t \le T)$ be a real valued, adapted RCLL-process. Set*

$$Y^I(t) \stackrel{\Delta}{=} e^{-\int_0^t \beta(s)\,ds} \left(\eta \vee \sup_{0 \le s \le t} \left\{ I(s) e^{\int_0^s \beta(s)\,ds} \right\} \right) \quad (0 \le t \le T)$$

and consider the rightcontinuous process of bounded variation C^I defined by

$$C^I(0-) \stackrel{\Delta}{=} 0, \quad dC^I(t) \stackrel{\Delta}{=} Y^I(t)\,dt + \beta(t)^{-1}\,dY^I(t) \quad (0 \le t \le T)\,.$$

This process is nondecreasing and adapted, and defines, therefore, a consumption plan, i.e., $C^I \in \mathcal{X}_+$. The level of satisfaction induced by this plan, $Y(C^I)$, coincides with Y^I and is minimal above I in the following sense:

$$Y(C^I)(t) = Y^I(t) = \min_{C \in \mathcal{X}_+,\, Y(C) \ge I} Y(C)(t) \quad \textit{for all} \quad 0 \le t \le T.$$

In particular, if $t \in [0, T]$ is a point of increase of C^I then $Y(C^I)(t) = I(t)$.

We say, the consumption plan C^I reflects its associated level of satisfaction at the process I.

Proof : Consider a consumption plan $C \in \mathcal{X}_+$. By Assumption 3.7, the process $A(C)$ defined by

$$A(C)(0-) \stackrel{\Delta}{=} \eta, \qquad A(C)(t) \stackrel{\Delta}{=} e^{\int_0^t \beta(s)\,ds} Y(C)(t) \quad (0 \le t \le T)$$

[3]A formal proof of this fact can be given by applying the Dynamic Programming Principle of Lemma 3.6 and by adapting Lemma 3.9 appropriately.

is increasing and adapted. In terms of $A(C)$, the restriction $Y(C) \geq I$ may be rewritten as

$$A(C)(t) \geq e^{\int_0^t \beta(s)\,ds} I(t) \quad \text{for all} \quad 0 \leq t \leq T\,.$$

Obviously, the minimal increasing process A^I which starts in $A^I(0-) \triangleq \eta$ and dominates the right side of this inequality is the running supremum

$$A^I(t) \triangleq \sup_{0 \leq s \leq t} \{\eta \vee e^{\int_0^s \beta(v)\,dv} I(s)\} \quad (0 \leq t \leq T).$$

Note that, by definition, $Y^I(t) = e^{-\int_0^t \beta(s)\,ds} A^I(t)$ and

$$dC^I(t) = \frac{1}{\beta(t)} e^{-\int_0^t \beta(s)\,ds}\, dA^I(t) \quad (0 \leq t \leq T)\,;$$

this yields $C \in \mathcal{X}_+$ and $Y(C^I) = Y^I$ as claimed. Furthermore, minimality of Y^I is inherited from the minimality of A^I. □

The above arguments suggests that, for a given Lagrange multiplier $M > 0$, there exists a canonical lower bound I at which the investor should optimally reflect his level of satisfaction. However, the heuristic way to construct this minimal level described above is far from being constructive, and, therefore, we would like to derive additional properties of this process that allow to characterize it more explicitly.

To this end, let us suppose that the felicity function u satisfies

$$\partial_y u(t, 0+) = +\infty \quad \text{for all } t \in [0, T].$$

Then our Kuhn-Tucker conditions imply that every t–Adam immediately starts consuming at time t: otherwise his level of satisfaction would remain zero over an open time interval, contradicting the inequality restriction

$$\phi_t(C_t^M)(v) \triangleq \mathbb{E}\left[\int_v^T \partial_y u\left(s, Y_t(C_t^M)(s)\right) \theta(s,v)\,ds \,\middle|\, \mathcal{F}_v\right] \leq M\psi(s) \quad (t \leq v \leq T)$$

for optimal plans. Hence, at time $v = t$, the first-order condition is binding for t–Adam and, therefore, we obtain the following equality for his level of satisfaction:

$$\phi_t(C_t^M)(t) = \mathbb{E}\left[\int_t^T \partial_y u(s, Y_t(s))\theta(s,t)\,ds \,\middle|\, \mathcal{F}_t\right] = M\psi(t)\,. \tag{16}$$

As pointed out above, we conjecture that t–Adam reflects his level of satisfaction at some lower bound I. Thus, Lemma 3.11 (adapted for initial time t and initial satisfaction zero) allows us to rewrite Equation (16) in terms of this process I:

$$\mathbb{E}\left[\int_t^T \partial_y u\left(s, \sup_{t \leq v \leq s}\left\{I(v) e^{-\int_v^s \beta(x)\,dx}\right\}\right) \theta(s,t)\,ds \,\middle|\, \mathcal{F}_t\right] = M\psi(t)\,. \tag{17}$$

Since I is a universal lower bound for *every* t–Adam's level of satisfaction, this equality should hold true for *every* $t \in [0, T)$. It yields the desired characterization of the 'canonical' minimal level of satisfaction as can be seen by the following

Theorem 3.12 *Suppose I is an adapted RCLL-process with $I(T) = 0$ which, for some fixed constant $M > 0$, satisfies Equation* (17) *at every time $t \in [0, T)$. Then the consumption plan C^I which reflects its associated level of satisfaction at I (cf. Lemma 3.11) is optimal for* (2) *given initial capital $w = \Psi(C^I)$. Moreover, its associated Lagrange multiplier is given by M.*

PROOF : We verify the first-order conditions for C^I:

$$\begin{aligned} \phi(C^I)(t) &= \mathbb{E}\left[\int_t^T \partial_y u\left(s, e^{-\int_0^s \beta(v)dv}\left(\eta \vee \sup_{0\le v\le s}\left\{I(v)e^{\int_0^v \beta(x)dx}\right\}\right)\right)\theta(s,t)ds\,\middle|\,\mathcal{F}_t\right] \\ &\le \mathbb{E}\left[\int_t^T \partial_y u\left(s, \sup_{t\le v\le s}\left\{I(v)e^{-\int_v^s \beta(x)\,dx}\right\}\right)\theta(s,t)\,ds\,\middle|\,\mathcal{F}_t\right] \qquad (18) \\ &= M\psi(t)\,. \end{aligned}$$

The first equality follows from the definition of C^I and the explicit formula for $Y(C^I)$ provided by Lemma 3.11. The last equality is precisely Equation (17). Hence, C^I satisfies the inequality constraint $\phi(C^I) \le M\psi$. Moreover, if, for fixed $\omega \in \Omega$, t is a point of increase for $C^I(\omega, .)$, then we have

$$Y(C^I)(\omega, s) = \sup_{t\le v\le s}\left\{I(\omega, v)e^{-\int_t^v \beta(x)\,dx}\right\}$$

for all $s \ge t$. Hence, we have equality in (18) whenever C^I is increasing. This proves the flat-off condition for C^I. □

Remark 3.13 *In a discrete time setting, it is easy to construct a solution to (the discrete-time analog of) Equation* (17) *via backwards induction, provided*

$$\partial_y u(t, 0+) = +\infty \quad \textit{and} \quad \partial_y u(t, +\infty) = 0$$

for all $t \in [0, T]$. In the present continuous-time framework, this procedure is no longer available. Indeed, the construction and even an existence proof of a solution to (17) *seem to be more involved and will, therefore, be discussed elsewhere.*

From the above results, we may derive the following method to construct explicit solutions to the utility maximization problem (2):

1. For every $M > 0$, find an adapted RCLL-process $I = I^M$ which solves Equation (17) for every $t \in [0, T)$.

2. For each $M > 0$, compute the price $\Psi(C^M)$ of the consumption plan $C^M \triangleq C^{I^M}$ which reflects its associated level of satisfaction at I^M.

3. The consumption plan C^M with $\Psi(C^M) = w$ is then the unique solution to the investor's utility maximization problem (2).

4 Explicit Solution in a Homogeneous Setting

In this section, we are going to derive some explicit solutions to the utility maximization problem (2) by applying the method described at the end of the preceding section.

4.1 Heuristics for Computing the Minimal Level of Satisfaction

Let us try to find a solution $I = (I(t), t \geq 0)$ to the minimal level equation (17) by looking for a 'good' candidate for this minimal level of satisfaction.

To this end, we first recall the structure of optimal consumption plans as they are derived in the 'classical' theory based on time-additive von Neumann-Morgenstern utility functionals. In such a setting, utility is obtained from the current *rate* of consumption, rather than from the instantaneous level of satisfaction. Applying methods of convex duality (confer, e.g., Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), one shows that the marginal felicity of an optimal consumption rate for this problem should equal some fixed multiple of the state-price density. This leads to the absolutely continuous optimal consumption plan $dC^{ac}(t) \equiv i(t, L\psi(t))\, dt$, where $i(t,.) \stackrel{\Delta}{=} (\partial_y u(t,.))^{-1}$ is the inverse of marginal felicity and L is a strictly positive constant.

Since, at least formally, the level of satisfaction $Y(C)$ plays the same role for our utility functional $U(C)$ as does the rate of consumption for the classic von Neumann-Morgenstern utilities, the above solution suggests to choose $C \in \mathcal{X}_+$ such that $Y(C)(t) \equiv i(t, L\psi(t))$. However, the right side of this equality will typically be of unbounded variation, while, under Assumption 3.7, the left side must have bounded variation for any choice of $C \in \mathcal{X}_+$. Hence, there might be no $C \in \mathcal{X}_+$ inducing a level of satisfaction of the form suggested above. In any case, however, we can consider the consumption plan C^L which reflects its associated level of satisfaction $Y^L \stackrel{\Delta}{=} Y(C^L)$ at

$$I^L(t) \stackrel{\Delta}{=} i(t, L\psi(t)) \quad (t \geq 0)\,.$$

This definition is illustrated by Figure 1.

Remark 4.1 *If we choose C^L as described, the last part of Lemma 3.11 shows that, for any two times s and t in which consumption occurs, the marginal rate of substitution, expressed in terms of the level of satisfaction, is equal to the economic rate of substitution:*

$$\frac{\partial_y u(s, Y^L(s))}{\partial_y u(t, Y^L(t))} = \frac{\psi(s)}{\psi(t)}\,.$$

Thus, I^L gives us a 'natural' candidate for the minimal level of satisfaction I we are looking for. However, there is an intuitive argument against the conjecture $I =$

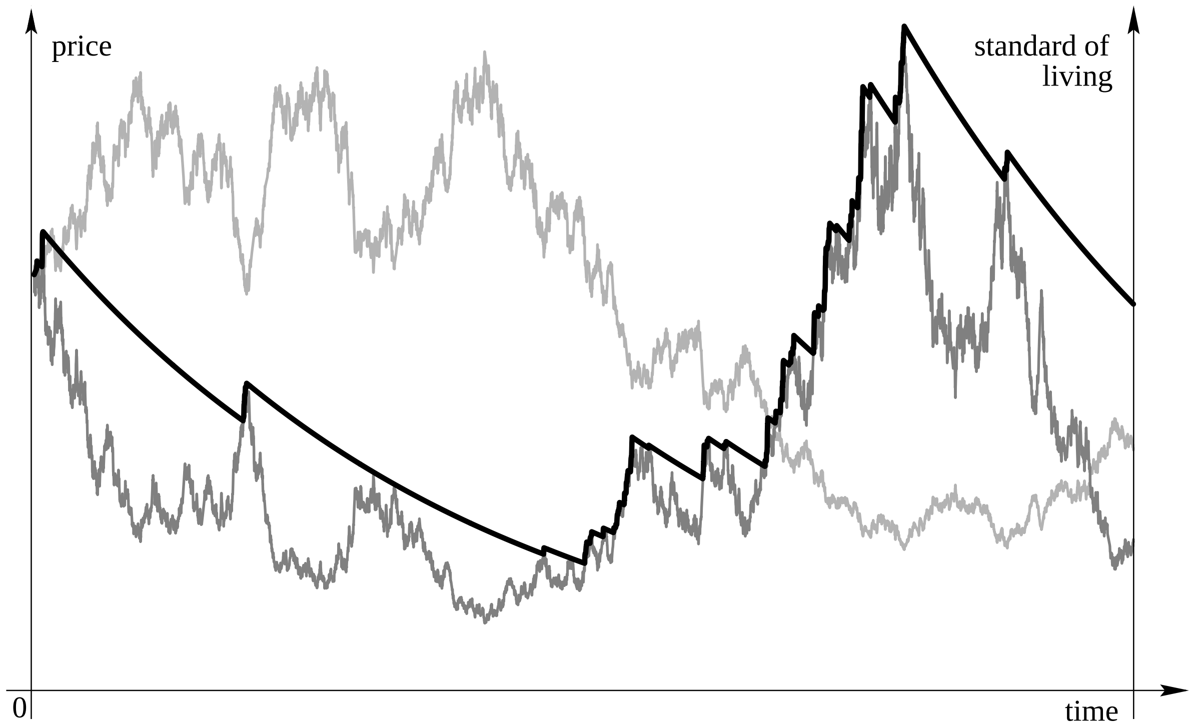

Figure 1: Typical paths for the state-price ψ (light grey line), the level of satisfaction $Y(C^L)$ (black line) and its minimal level I^L (grey line).

I^L for some $L = L(M) > 0$. Indeed, consider Equation (17), which characterizes the minimal level of satisfaction, for times t close to the time horizon T. It is easy to see that a solution I to this equation must converge to zero as $t \uparrow T$. This corresponds to the observation that, in the Hindy-Huang-Kreps-setting, the contribution to total expected utility is considerably smaller for consumption at times close to the time horizon T than for consumption at time zero, say. Our candidate $i(t, M\psi(t))$, however, converges to zero as $t \uparrow T$ iff $\psi(t) \uparrow +\infty$, a condition clearly *not* fulfilled by any typical state-price deflator. Therefore, the received candidate plans C^L do not take into account properly the time horizon T. In particular, the described minimal level will not induce optimal consumption plans, if — as we assumed up to now — the time horizon is finite. But for an investor with *infinite time horizon* this caveat does not apply. In fact, we can prove optimality of our candidate policies in a homogeneous setting under this condition; confer Corollary 4.8.

4.2 The Homogeneous Setting

In contrast to the preceding sections let us henceforth assume that the investor's time horizon is infinite: $T = +\infty$. We furthermore suppose his felicity function to have the separable, homogeneous form

$$u(t,y) = e^{-\delta t}\frac{1}{\alpha}y^{\alpha} \quad (t \geq 0, y \geq 0) \tag{19}$$

for some constant $\alpha \in (-\infty, 1)\backslash\{0\}$ and denote by

$$i(t,z) \triangleq \left(e^{\delta t}z\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0, z > 0)$$

the inverse of its associated marginal felicity function $\partial_y u(t,.)$.

Remark 4.2 *The case $\alpha = 0$, corresponding to '*log*-felicity', can be treated with the same method as the 'power-felicities' above. However, for ease of exposition, we leave this case to the interested reader.*

We assume the function $\beta(.)$ of Assumption 3.7 to be a strictly positive constant $\beta(.) \equiv \beta > 0$. Hence, the level of satisfaction is a time-homogenous, exponentially weighted average of past consumption:

$$Y(C)(t) = \eta e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)}\,dC(s)$$

with constants $\beta > 0$ and $\eta \geq 0$.

To ensure that the utility maximization problem is well-posed also for an infinite time horizon $T = +\infty$, it is necessary (not sufficient, see Theorem 4.9 below) to introduce

Assumption 4.3 $\delta + \alpha\beta > 0$.

Indeed, in case $\delta + \alpha\beta \leq 0$, it its easy to see that the investor obtains infinite utility $\mathbb{E}U(C) = +\infty$ by consuming all his wealth in one single gulp at time $t = 0$.

Furthermore, we assume that the unique state-price density ψ is of the form

$$\psi(t) = \exp\left(-\theta X(t) - (r + \pi(-\theta))t\right) \quad (t \geq 0),$$

for some Lévy process X with finite Laplace-exponent $\pi(\xi)$ $(\xi \in \mathbb{R})$ (under $\mathbb{P}$). Hence, interest rates are constant, $r(t) \equiv r \geq 0$, and uncertainty is introduced by a stochastic process X with stationary and independent increments which possesses all exponential moments

$$\mathbb{E}\exp\left(\xi X(t)\right) < +\infty \quad (\xi \in \mathbb{R},\ t \geq 0)\,.$$

The Laplace-exponent $\pi(.)$ of X is then defined via

$$\mathbb{E}\exp\left(\xi X(t)\right) = \exp(\pi(\xi)t) \quad \text{for all } \xi \in \mathbb{R},\, t \geq 0\,;$$

see, e.g., Bertoin (1996). The constant $\theta > 0$ can be viewed as the 'market price of risk'. The deterministic case $X(t) \equiv \text{const} \cdot t$ is treated in Hindy, Huang, and Kreps (1992) and Bank and Riedel (1998), and will, therefore, be excluded implicitly in the following.

Example 4.4 *1. For $X = (W(t),\ t \geq 0)$, a standard Brownian motion, we have $\pi(\xi) = \frac{1}{2}\xi^2$, and the state-price density*

$$\psi(t) = \exp\left(-\theta W(t) - \left(r + \tfrac{1}{2}\theta^2\right) t\right) \quad (t \geq 0)$$

takes the well known form of a geometric Brownian motion. This specification of ψ corresponds to the setup studied in Hindy and Huang (1993).

2. If $X = (\pm N(t),\ t \geq 0)$ is a Poisson process with upward (downward) jumps and intensity λ, then $\pi(\xi) = \lambda(e^{\pm\xi} - 1)$ and, therefore,

$$\psi(t) = \exp\left(\mp\theta N(t) - \left(r + \lambda(e^{\mp\theta} - 1)\right) t\right) \quad (t \geq 0)$$

is a geometric Poisson process.

Remark 4.5 *Note that the above examples describe indeed complete financial markets if $\mathbb{F}$ is the augmented filtration generated by X.*

In the above setting, the consumption plans C^L $(L > 0)$ defined in the preceding section can be represented in the following form:

$$dC^L(t) = \frac{1}{\beta} e^{-\beta t}\, dA^L(t) \quad (t \geq 0)$$

where, for $t \geq 0$,

(20) $$A^L(0-) \stackrel{\Delta}{=} \eta, \quad A^L(t) \stackrel{\Delta}{=} \eta \vee \left\{L^{-\frac{1}{1-\alpha}} S(t)\right\}$$

with

(21) $$S(t) \stackrel{\Delta}{=} \sup_{0 \leq s \leq t} \left\{\psi(s)^{-\frac{1}{1-\alpha}} e^{\left(\beta - \frac{\delta}{1-\alpha}\right)s}\right\}.$$

As usual we have

$$Y(C^L)(t) = e^{-\beta t} A^L(t) \quad (t \geq 0).$$

4.3 A Solution to the Level Equation

Let us now prove that the consumption plans C^L described above are indeed optimal in their respective class $\mathcal{A}(w)(\Psi(C^L))$. Following the method proposed at the end of Section 3.2, we first show that, in the above homogeneous setting, the candidates I^L $(L > 0)$ for the minimal level of satisfaction do indeed solve Equation (17) for some $M = M(L) > 0$:

Lemma 4.6 *In the homogeneous setting of Section 4.2, the process*

$$I^L(t) \stackrel{\Delta}{=} i(t, L\psi(t)) = \left(Le^{\delta t}\psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

solves Equation (17) *for*

(22) $$M \stackrel{\Delta}{=} \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)s} \inf_{0 \leq v \leq s} \{e^{-(\beta(1-\alpha)-\delta)v} \psi(v)\}\, ds\right] L < +\infty.$$

PROOF : We have

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^{+\infty} \partial_y u\left(s, e^{-\beta s} \sup_{t \le v \le s}\{I^L(v)e^{\beta v}\}\right) \beta e^{-\beta(s-t)}\, ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{+\infty} \beta e^{\beta t} e^{-(\delta+\alpha\beta)s} \inf_{t \le v \le s}\{L e^{-(\beta(1-\alpha)-\delta)v}\psi(v)\}\, ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)s} \inf_{0 \le v \le s}\{L e^{-(\beta(1-\alpha)-\delta)v} \frac{\psi(t+v)}{\psi(t)}\}\, ds \,\middle|\, \mathcal{F}_t\right]\psi(t) \\
&= \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)s} \inf_{0 \le v \le s}\{L e^{-(\beta(1-\alpha)-\delta)v}\psi(v)\}\, ds\right]\psi(t)
\end{aligned}
$$

where the last equation holds true because X is a Lévy process. Thus, I^L does indeed solve Equation (17) for $M = M(L) > 0$ as defined in (22). Note that $M < +\infty$ because the infimum in its definition is always less than or equal to 1 and because $\delta + \alpha\beta > 0$ by Assumption 4.3. □

By the same arguments as in the proof of Theorem 3.12, we now can show that the consumption plans C^L satisfy the first-order conditions (ii) and (iii) with $T = +\infty$. Moreover, it is easy to see that these conditions are sufficient for optimality also in the infinite horizon case:

Theorem 4.7 *A consumption plan C^* solves* (2) *for $T = +\infty$ if the following conditions hold true for some Lagrange multiplier $M \ge 0$:*

(i) $\Psi(C^*) = w$,

(ii) $\phi(C^*) \le M\psi$ $\mathbb{P}$*-a.s.,*

(iii) C^* *is flat off* $\{\phi(C^*) = M\psi\}$ $\mathbb{P}$*-a.s., i.e.,*

$$\mathbb{E}\int_0^{+\infty} 1_{\{\phi(C^*)(t) \ne M\psi(t)\}}\, dC^*(t) = 0$$

where $\phi(C)$ is defined for $C \in \mathcal{X}_+$ by (5) *with $T = +\infty$.*

PROOF : Without loss of generality we may assume $\mathbb{E}U(C^*) < +\infty$. Observe now that every expression in the argument for the sufficiency part of Theorem 3.3 is well-defined also for $T \triangleq +\infty$. Hence, we may use this argument to deduce that indeed $\mathbb{E}U(C^*) \ge \mathbb{E}U(C)$ for any other consumption plan $C \in \mathcal{A}(w)$. □

We finally obtain

Corollary 4.8 *In the homogeneous setting described in Section 4.2, the consumption plan C^L is optimal given initial capital $w = \Psi(C^L)$, provided this value is finite.*

4.4 Prices and Utilities

The preceding section shows that — in the homogeneous setting of Section 4.2 — the consumption plans C^L $(L > 0)$ are optimal in their respective class *provided their price is finite*. Hence, we still have to check for which parameter values of the problem this condition is satisfied. Furthermore, we should calculate the exact prices for varying Lagrange multiplier $M > 0$ in order to find the plan whose price coincides with a given initial capital $w > 0$.

4.4.1 Well-Posedness of the Utility Maximization Problem

We show that, in our homogeneous framework, the well-posedness of problem (2) is (essentially) equivalent to the finiteness of all prices of our candidate policies C^L $(L > 0)$. Thus, our method yields the complete solution to the problem provided this problem is well-posed.

Theorem 4.9 *The following assertions are equivalent:*

(i) Finite prices: $\Psi(C^L) < +\infty$ *for some (all)* $L > 0$*.*

(ii) Finite utilities: $\mathbb{E}U(C^L) < +\infty$ *for some (all)* $L > 0$*.*

(iii) The investor's rate of time preference δ *satisfies*

$$\delta > \hat{\delta} \triangleq \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta)\,. \tag{23}$$

If $\delta < \hat{\delta}$*, then, for any initial wealth* $w > 0$*, there is a budget-feasible* C *with infinite expected utility, i.e., the optimization problem* (2) *is ill-posed.*

Remark 4.10 *Note that there is a slight 'gap' in Theorem 4.9, since it leaves open whether or not the optimization problem is well-posed in case* $\delta = \hat{\delta}$*. However, as shown by Lemma 4.15 below, this case can be treated under some additional assumption.*

The proof of Theorem 4.9 will be prepared by the following Lemmata 4.11, 4.12, and 4.13.

Lemma 4.11 *(i) In terms of the increasing process* A^L*, we may express the price of the consumption plan* C^L *as*

$$\Psi^L \triangleq \Psi(C^L) = \tfrac{1}{\beta}\left(\mathbb{E}^* A^L(\tau^*) - \eta\right) \quad (L > 0)\,, \tag{24}$$

where τ^* *is an independent exponential random time with parameter* $r + \beta > 0$*.*

(ii) We have $\Psi^L < +\infty$ for some (and then all) $L > 0$ iff

$$\mathbb{E}^* S(\tau^*) < \infty\,, \tag{25}$$

where τ^* is as in (i). In particular, the price of every *policy C^L $(L > 0)$ is finite if just one of these prices is finite.*

(iii) The mapping $L \mapsto \Psi^L$ is nonnegative, nondecreasing, and convex. If prices are finite, we have $\Psi^L \uparrow +\infty$ as $L \downarrow 0$ and $\Psi^L \downarrow 0$ as $L \uparrow +\infty$. In particular, for every initial capital $w > 0$ there is a consumption policy C^L with price $\Psi^L = w$ in this case.

PROOF: From $dC^L(t) = \frac{1}{\beta}e^{-\beta t}\,dA^L(t)$ and partial integration of the price functional, we deduce for all $L > 0$

$$\Psi^L = \frac{1}{\beta}\mathbb{E}^* \lim_{T\uparrow+\infty}\left(A^L(T)e^{-(r+\beta)T} - \eta + \int_0^T A^L(t)(r+\beta)e^{-(r+\beta)t}\,dt\right). \tag{26}$$

Hence,

$$\mathbb{E}^* A^L(\tau^*) = \mathbb{E}^* \int_0^{+\infty} A^L(t)(r+\beta)e^{-(r+\beta)t}\,dt < +\infty \tag{27}$$

is necessary for $\Psi^L < +\infty$. It is also sufficient since it implies

$$\lim_{T\uparrow+\infty} A^L(T)e^{-(r+\beta)T} = 0 \quad \mathbb{P}^*\text{-a.s.} \tag{28}$$

Indeed, otherwise we have $\limsup_{T\uparrow+\infty} A^L(T)e^{-(r+\beta)T} > 0$ with positive $\mathbb{P}^*$-probability. Thus, on a set with positive $\mathbb{P}^*$-measure, there is a random $\epsilon > 0$ such that

$$A^L(\sigma_n)e^{-(r+\beta)\sigma_n} \geq \epsilon$$

along a sequence of random times σ_n tending to $+\infty$ as $n \uparrow +\infty$. Without loss of generality we may assume that $\sigma_{n+1} - \sigma_n \geq 1$ for all n. Since A^L is nondecreasing we have $A^L(t)e^{-(r+\beta)t} \geq \epsilon e^{-(r+\beta)} > 0$ whenever $t \in [\sigma_n, \sigma_n + 1)$ for some n. This implies $\int_0^{+\infty} A^L(t)(r+\beta)e^{-(r+\beta)t}\,dt = +\infty$ with positive $\mathbb{P}^*$-probability. Hence, (27) implies (28). Furthermore the preceding considerations yield that (i) is implied by (26).

For (ii) it remains to note that $\mathbb{E}^* A^L(\tau^*) < +\infty$ is equivalent to $\mathbb{E}^* S(\tau^*) < +\infty$. This follows from $A^L(\tau^*) \geq L^{-\frac{1}{1-\alpha}} S(\tau^*)$ and

$$\mathbb{E}^* A^L(\tau^*) = L^{-\frac{1}{1-\alpha}}\mathbb{E}^* S(\tau^*) + \mathbb{E}^*\left(\eta - L^{-\frac{1}{1-\alpha}} S(\tau^*)\right)^+ \leq L^{-\frac{1}{1-\alpha}}\mathbb{E}^* S(\tau^*) + \eta\,.$$

From (i) we deduce that Ψ^L is nonnegative, nondecreasing, and convex, since so is A^L. If prices are finite, $A^{L_0}(\tau^*)$ is $\mathbb{P}^*$-integrable. Thus, $\Psi^L \downarrow 0$ for $L \uparrow +\infty$ by dominated convergence. For $L \downarrow 0$, we have $\Psi^L \geq \Delta C^L(0) \uparrow +\infty$. This yields (iii). □

The following is an analog of Lemma 4.11 for utilities instead of prices:

Lemma 4.12 *(i) In terms of the increasing process A^L, we may express the expected utility of plan C^L as*

$$\mathbb{E}U(C^L) = \frac{1}{\alpha(\delta + \alpha\beta)}\mathbb{E}\left(A^L(\tau)\right)^{\alpha} \quad (L > 0)\,, \tag{29}$$

where τ is an independent exponential random time with parameter $\delta + \alpha\beta$.

(ii) We have $\mathbb{E}U(C^L) < +\infty$ for all $L > 0$ iff

$$\mathbb{E}\,(S(\tau))^{\alpha} < \infty\,, \tag{30}$$

where τ is as in (i). In particular, the expected utility of every *policy C^L is finite if just one of these utilities is finite.*

PROOF : Note first that, because of Assumption 4.3, we have $\delta + \alpha\beta > 0$, and, therefore, τ is well-defined. Now, (i) follows from $Y(C^L)(t) = e^{-\beta t}A^L(t)$ and the definition of the utility functional $U(.)$. For (ii) we note that $\left(A^L(\tau)\right)^{\alpha} \geq L^{-\frac{\alpha}{1-\alpha}}(S(\tau))^{\alpha}$ and

$$\mathbb{E}\left(A^L(\tau)\right)^{\alpha} \leq \eta^{\alpha} + L^{-\frac{\alpha}{1-\alpha}}\mathbb{E}\,(S(\tau))^{\alpha}\,.$$

□

Lemma 4.11 and Lemma 4.12 are valid for any semimartingale state-price density which induces a constant interest rate. Yet, for the following lemma we need the special Lévy-structure of ψ.

Lemma 4.13 *(i) Let σ be an exponential random time independent of X. We have the Wiener-Hopf factorization*

$$\mathbb{E}\exp\left(\xi \sup_{0\leq s\leq\sigma} X(s)\right)\mathbb{E}\exp\left(\xi \inf_{0\leq s\leq\sigma} X(s)\right) = \mathbb{E}\exp\left(\xi X(\sigma)\right) \tag{31}$$

for all $\xi \in \mathbb{R}$.

(ii) If, in addition, X has no positive jumps and is neither a deterministic drift nor the negative of a subordinator, then $\sup_{0\leq s\leq\sigma} X(s)$ is exponentially distributed. The parameter ζ of its distribution is uniquely determined by $\pi(\zeta) = \xi$, where ξ is the parameter of the exponential distribution of σ.

(iii) Under the risk-neutral measure $\mathbb{P}^*$ induced by ψ, X is again a Lévy-process with finite exponential moments and its $\mathbb{P}^*$-Laplace exponent is given by

$$\pi^*(\xi) = \pi(\xi - \theta) - \pi(-\theta). \tag{32}$$

PROOF :

(i) For $t \geq 0$, let $\tilde{X}(t) \overset{\Delta}{=} \sup_{0\leq s\leq t} X(s)$. By Theorem VI.5(i) in Bertoin (1996), the random variables $\tilde{X}(\sigma)$ and $\tilde{X}(\sigma) - X(\sigma)$ are independent. Hence,

$$\begin{aligned} \mathbb{E}\exp(\xi X(\sigma)) &= \mathbb{E}\left[\exp(\xi\tilde{X}(\sigma))\exp(-\xi(\tilde{X}(\sigma) - X(\sigma)))\right] \\ &= \mathbb{E}\exp(\xi\tilde{X}(\sigma))\mathbb{E}\exp(-\xi(\tilde{X}(\sigma) - X(\sigma))) . \end{aligned} \tag{33}$$

Using the Duality Lemma II.2 in Bertoin (1996) and the independence of X and σ, we see that

$$\tilde{X}(\sigma) - X(\sigma) = \sup_{0\leq s\leq\sigma} \{X((\sigma - s)-) - X(\sigma)\}$$

has the same law as

$$\sup_{0\leq s\leq\tau} \{-X(s)\} = -\inf_{0\leq s\leq\tau} X(s) .$$

In connection with Equation (33), this yields (i).

(ii) This is Corollary VII.1.2 in Bertoin (1996).

(iii) By definition of ψ, the density process Z for $\mathbb{P}$ and $\mathbb{P}^*$ is given by

$$Z(t) \overset{\Delta}{=} \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp(-\theta X(t) - \pi(-\theta)t) \quad (t \geq 0) .$$

Hence, for $s, t \geq 0$, we may calculate the conditional $\mathbb{P}^*$-Laplace transform of the increment $X(t+s) - X(t)$ given $\mathcal{F}_t$ as follows:

$$\begin{aligned} &\mathbb{E}^*\left[\exp(\xi(X(t+s) - X(t)))\,|\,\mathcal{F}_t\right] \\ &= \frac{1}{Z(t)}\mathbb{E}\left[\exp(\xi(X(t+s) - X(t)))Z(t+s)\,|\,\mathcal{F}_t\right] \\ &= \frac{1}{Z(t)}\mathbb{E}\left[\exp((\xi-\theta)(X(t+s) - X(t)))\,|\,\mathcal{F}_t\right]\exp(-\theta X(t) - \pi(-\theta)(t+s)) \\ &= \exp(s(\pi(\xi-\theta) - \pi(-\theta))) . \end{aligned}$$

Since the last quantity is deterministic and does not depend on t, the above calculation shows that, also under $\mathbb{P}^*$, X has independent and stationary increments. Furthermore, we can easily read off Equality (32) for the $\mathbb{P}^*$-Laplace exponent $\pi^*(.)$.

□

Now, we are in the position to give the

Proof of Theorem 4.9

(i)⇔(iii) By Lemma 4.11 (ii), we know that (i) is equivalent to

$$\begin{aligned}
\infty &> \mathbb{E}^* S(\tau^*) \\
&= \mathbb{E}^* \sup_{0\le s\le\tau^*} \exp\left(\frac{\theta}{1-\alpha}X(s) + \left(\frac{\pi(-\theta)+r+\beta(1-\alpha)-\delta}{1-\alpha}\right)s\right) \\
&= \mathbb{E}^* \exp\left(\frac{\theta}{1-\alpha}\sup_{0\le s\le\tau^*}\{X(s)+\mu s\}\right)
\end{aligned}$$

where τ^* is an independent exponential random time with parameter $r+\beta$ and

$$\mu \triangleq \frac{1}{\theta}(\pi(-\theta)+r+\beta(1-\alpha)-\delta).$$

By Lemma 4.13 (i), the above calculation yields that (i) holds true iff

$$\mathbb{E}^* \exp\left(\frac{\theta}{1-\alpha}\{X(\tau^*)+\mu\tau^*\}\right) < +\infty.$$

Since τ^* is independent of X and exponentially distributed with parameter $r+\beta$, we may use Fubini's Theorem to obtain equivalence of (i) and

$$r+\beta > \pi^*\left(\frac{\theta}{1-\alpha}\right) + \frac{\theta\mu}{1-\alpha}.$$

Using the transformation rule (32), it is easy to see that this condition is indeed equivalent to (iii).

(ii)⇔(iii) Making use of Lemma 4.12, we may follow a similar line of arguments as in the proof of (ii)⇔(iii). First, (ii) is equivalent to

$$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha}\{X(\tau)+\mu\tau\}\right) < +\infty$$

where τ is an independent exponential random time with parameter $\delta+\alpha\beta > 0$, and where μ is defined as above. Using Fubini's Theorem allows us to conclude the equivalence of (ii) and

$$\delta+\alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\theta\mu}{1-\alpha},$$

which, by an easy calculation, can be shown to be equivalent to (iii), too.

To prove that problem (2) is ill-posed in case $\delta < \hat{\delta}$, let $\bar{\delta} > \hat{\delta}$ be given and consider the consumption plan $\bar{C}^L$ obtained from reflecting the level of satisfaction at the lower bound process

$$\bar{l}^L(t) \triangleq \left(Le^{\bar{\delta}t}\psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \ge 0).$$

The corresponding increasing process $\bar{A}^L$ is given by

$$\bar{A}^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0\le s\le t} \{X(s) + \bar{\mu}s\}\right) \quad (t \ge 0)$$

where

$$\bar{\mu} \stackrel{\Delta}{=} \frac{1}{\theta}\left(\pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta}\right).$$

From (i)$\Leftrightarrow$(iii) we know that the price of every policy $\bar{C}^L$ is finite because $\bar{\delta} > \hat{\delta}$. Hence, for any initial wealth $w > 0$, we can find $L = L(w)$ such that $\bar{C}^{L(w)}$ is budget-feasible.

By the same arguments as in the proof of Lemma 4.12, one can now show that the expected utility of the plan $\bar{C}^{L(w)}$ is finite iff

$$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha} \sup_{0\le s\le \tau} \{X(s) + \bar{\mu}s\}\right) < \infty$$

where τ is, as usual, an independent exponential random time with parameter $\delta + \alpha\beta$. From equality (31), we deduce that the above relation holds true iff

$$\mathbb{E}\exp\left(\frac{\alpha\theta}{1-\alpha}\{X(\tau) + \bar{\mu}\tau\}\right) < \infty.$$

Since τ is independent of X and exponentially distributed, this is equivalent to

$$\delta + \alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\theta\bar{\mu}}{1-\alpha}. \tag{34}$$

Now, note that, for $\bar{\delta} \downarrow \hat{\delta}$, the right side of this inequality increases to

$$\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\theta\hat{\mu}}{1-\alpha} = \hat{\delta} + \alpha\beta > \delta + \alpha\beta, \tag{35}$$

where

$$\hat{\mu} \stackrel{\Delta}{=} \lim_{\bar{\delta}\downarrow\hat{\delta}} \bar{\mu} = \frac{1}{\theta}\left(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta}\right).$$

Equation (35) follows by the definition of $\hat{\delta}$. Hence, there are $\bar{\delta} > \hat{\delta}$ for which Inequality (34) is violated and for which, therefore, the associated plans $\bar{C}^L$ have infinite expected utility, even though their price is finite. □

4.4.2 Explicit Computations

In order to obtain closed-form solutions for (2), it still remains to calculate all prices $\Psi(C^L)$ $(L > 0)$ and to identify the parameter $L(w)$ for which $\Psi(C^{L(w)}) = w$. To this end, let us introduce the Lévy process

$$Z(t) \stackrel{\Delta}{=} X(t) + \mu t \quad (t \ge 0),$$

where

$$\mu \stackrel{\Delta}{=} \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \delta)\,. \tag{36}$$

This allows us to rewrite A^L in the form

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha}\tilde{Z}(t)\right) \quad (t \geq 0)$$

where

$$\tilde{Z}(t) \stackrel{\Delta}{=} \sup_{0\leq s\leq t} Z(s) = \sup_{0\leq s\leq t} \{X(s) + \mu s\} \quad (t \geq 0)\,.$$

Now, we are able to compute the prices $\Psi(C^L)$ and utilities $\mathbb{E}U(C^L)$ $(L > 0)$ explicitly in the following two cases:

Assumption 4.14 *(i) Z has decreasing paths only.*

(ii) Z does not *have decreasing paths only and all its jumps are nonpositive ($\Delta Z \leq 0$).*

Let τ and τ^* be exponential random times, independent of X with parameter $\delta + \alpha\beta$ and $r + \beta > 0$, respectively. Then, both Assumption 4.14 (i) and Assumption 4.14 (ii) ensure that $\tilde{Z}(\tau)$ and $\tilde{Z}(\tau^*)$ are exponentially distributed under $\mathbb{P}$ and $\mathbb{P}^*$ respectively. In fact, if Assumption 4.14 (i) holds true, we evidently have $\tilde{Z}(t) \equiv 0$ which corresponds to the parameter values $\zeta = \zeta^* = 0$ for the respective exponential distribution. Under Assumption 4.14 (ii), we may apply Lemma 4.13 (ii) with Z instead of X to identify the exponential parameters as the unique positive solutions to

$$\pi(\zeta) + \mu\zeta = \delta + \alpha\beta \quad \text{and} \quad \pi^*(\zeta^*) + \mu\zeta^* = r + \beta \tag{37}$$

respectively.

Thus, proceeding from Equations (24) and (29), we now can compute

$$\Psi(C^L) = \frac{1}{\beta} \cdot \begin{cases} \left(L^{-\frac{1}{1-\alpha}} - \eta\right)^+ & \text{if } \zeta^* = 0 \\ \dfrac{(1-\alpha)\zeta^*}{(1-\alpha)\zeta^* - \theta} L^{-\frac{1}{1-\alpha}} - \eta & \text{if } \eta \leq L^{-\frac{1}{1-\alpha}},\ \zeta^* > 0 \\ \dfrac{\theta}{(1-\alpha)\zeta^* - \theta} \eta^{-\frac{(1-\alpha)\zeta^* - \theta}{\theta}} L^{-\frac{\zeta^*}{\theta}} & \text{else} \end{cases} \tag{38}$$

and

$$\mathbb{E}U(C^L) = \frac{1}{\alpha(\delta + \alpha\beta)} \cdot \begin{cases} \eta^\alpha \vee L^{-\frac{\alpha}{1-\alpha}} & \text{if } \zeta = 0 \\ \dfrac{(1-\alpha)\zeta}{(1-\alpha)\zeta - \alpha\theta} L^{-\frac{\alpha}{1-\alpha}} & \text{if } \eta \leq L^{-\frac{1}{1-\alpha}},\ \zeta > 0 \\ \eta^\alpha + \dfrac{\alpha\theta}{(1-\alpha)\zeta - \alpha\theta} \eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\theta}} L^{-\frac{\zeta}{\theta}} & \text{else.} \end{cases}$$

Hence, an agent with initial wealth $w > 0$ optimally follows the consumption plan $C^{L(w)}$ with

$$L(w) \triangleq \begin{cases} (\beta w + \eta)^{-(1-\alpha)} & \text{if } \zeta^* = 0 \\ \left(\frac{(1-\alpha)\zeta^* - \theta}{(1-\alpha)\zeta^*} (\beta w + \eta) \right)^{-(1-\alpha)} & \text{if } w \geq \hat{w},\ \zeta^* > 0 \\ \left(\frac{(1-\alpha)\zeta^* - \theta}{\theta} \eta^{\frac{(1-\alpha)\zeta^* - \theta}{\theta}} \beta w \right)^{-\frac{\theta}{\zeta^*}} & \text{else} \end{cases}$$

where $\hat{w} \triangleq \frac{1}{\beta} \frac{\theta}{(1-\alpha)\zeta^* - \theta} \eta$.

Furthermore, using Lemma 4.13 (iii), one can show that $\zeta^* = \zeta + \theta$ by a straightforward calculation. This allows us to represent the agent's maximal utility (the value $v(w)$ of the program (2)) by

$$v(w) = \frac{1}{\alpha(\delta + \alpha\beta)} \cdot \begin{cases} (\beta w + \eta)^{\alpha} & \text{if } \zeta^* = 0 \\ \zeta \left(\frac{1-\alpha}{(1-\alpha)\zeta - \alpha\theta} \right)^{1-\alpha} \left(\frac{\beta w + \eta}{\zeta + \theta} \right)^{\alpha} & \text{if } w \geq \hat{w},\ \zeta^* > 0 \\ \eta^{\alpha} + \alpha \eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\zeta + \theta}} \left(\frac{\theta \beta w}{(1-\alpha)\zeta - \alpha\theta} \right)^{\frac{\zeta}{\zeta+\theta}} & \text{else.} \end{cases}$$

The above formulae give us the desired explicit solution to the investor's utility maximization problem (2) in the homogeneous setting of Section 4.2.

As pointed out by Remark 4.10, Theorem 4.9 does not characterize completely the parameter values for which problem (2) is well-posed in the present context. However, under Assumption 4.14, this problem can be solved:

Lemma 4.15 *Under Assumption 4.14, the parameter restriction $\delta > \hat{\delta}$ of Lemma 4.9 (iii) is also necessary for problem* (2) *to be well-posed. More precisely, suppose that Assumption 4.14 is satisfied and that the parameters of the problem are such that*

$$\delta \leq \hat{\delta} = \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta). \tag{39}$$

Then we have

$$\sup_{C \in \mathcal{A}(w)} \mathbb{E} U(C) = +\infty$$

for any initial capital $w > 0$.

Proof : As in the proof of Theorem 4.9, choose some $\bar{\delta} > \hat{\delta}$ and consider, for every $L > 0$, the lower bound $\bar{I}^L$ obtained from I^L by replacing δ with $\bar{\delta}$. Again, the corresponding consumption plans will be denoted by $\bar{C}^L$, and we will write $\bar{S}$ for the analog of the supremum process S. For simplicity, we assume that $\eta = 0$.

We have

$$\Psi(\bar{C}^L) = \frac{L^{-\frac{1}{1-\alpha}}}{\beta} \mathbb{E}^* \bar{S}(\tau^*)$$

and

$$\mathbb{E}U(\bar{C}^L) \geq \mathbb{E}\int_0^\infty e^{-\hat{\delta}t}\frac{1}{\alpha}\left(e^{-\beta t}L^{-\frac{1}{1-\alpha}}\bar{S}(t)\right)^\alpha\,dt = \frac{L^{-\frac{1}{1-\alpha}}}{\alpha(\delta+\alpha\beta)}\mathbb{E}\bar{S}(\tau)^\alpha$$

where τ^* and τ are independent exponential random times with parameters $r+\beta > 0$ and $\hat{\delta}+\alpha\beta > 0$ respectively.

In order to meet the budget-constraint, we choose L such that $\Psi(\bar{C}^L) = w$. Note that this is indeed possible because of $\bar{\delta} > \hat{\delta}$. By the above calculations, this gives us

$$v(w) \triangleq \sup_{C\in\mathcal{A}(w)} \mathbb{E}U(C) \geq \frac{(\beta w)^\alpha}{\alpha(\delta+\alpha\beta)}\frac{\mathbb{E}\bar{S}(\tau)^\alpha}{(\mathbb{E}^*\bar{S}(\tau^*))^\alpha}.$$

Hence, to show that $v(w) \equiv +\infty$, it suffices to prove

$$\frac{\mathbb{E}\bar{S}(\tau)^\alpha}{(\mathbb{E}^*\bar{S}(\tau^*))^\alpha} \to +\infty \quad \text{as } \bar{\delta} \downarrow \hat{\delta}. \tag{40}$$

Using Lemma 4.13 (ii), it is easy to see that

$$\mathbb{E}\bar{S}(\tau)^\alpha = \frac{\bar{\zeta}}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} \quad \text{and} \quad \mathbb{E}^*\bar{S}(\tau^*) = \frac{\bar{\zeta}^*}{\frac{\theta}{1-\alpha} - \bar{\zeta}^*}$$

where $\bar{\zeta}$ and $\bar{\zeta}^*$ are determined by

$$\pi(\bar{\zeta}) + \bar{\mu}\bar{\zeta} = \hat{\delta} + \alpha\beta \quad \text{and} \quad \pi^*(\bar{\zeta}^*) + \bar{\mu}\bar{\zeta}^* = r + \beta \tag{41}$$

with $\bar{\mu} \triangleq \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta})$. A straightforward calculation based on Lemma 4.13 (iii) shows that $\bar{\zeta}^* = \bar{\zeta} + \theta$ by a straightforward calculation.

This allows us to conclude that

$$\frac{\mathbb{E}\bar{S}(\tau)^\alpha}{(\mathbb{E}^*\bar{S}(\tau^*))^\alpha} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha}\frac{(\frac{\theta}{1-\alpha} - \bar{\zeta}^*)^\alpha}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha}\left(\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}\right)^{-(1-\alpha)}. \tag{42}$$

Using the definition of $\hat{\delta}$, one can show that $\hat{\zeta} \triangleq \frac{\alpha\theta}{1-\alpha}$ is the unique solution to

$$\pi(\hat{\zeta}) + \hat{\mu}\hat{\zeta} = \hat{\delta} + \alpha\beta \tag{43}$$

with $\hat{\mu} \triangleq \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta})$. Since, by definition, $\bar{\zeta}$ depends continuously on $\bar{\delta}$ and because Equation (43) is the limit of Equation (41) for $\bar{\delta} \downarrow \hat{\delta}$, this shows that $\bar{\zeta} \to \hat{\zeta} = \frac{\alpha\theta}{1-\alpha}$ as $\bar{\delta} \downarrow \hat{\delta}$. Now, the claimed convergence (40) can be read off Equation (42). □

Remark 4.16 *Theorem 4.9 and Lemma 4.15 show that our method provides the complete solution to the utility maximization problem* (2)*.*

4.5 Examples: Brownian Motion and Poisson Process

This section illustrates the preceding results by two case studies, X being either a Brownian motion or a Poisson process. For the special case of Brownian motion, our results allow to recover (and extend) the results by Hindy and Huang (1993). In particular, we will recover the singularity of optimal consumption plans with respect to Lebesgue measure in this setting. By contrast, in the Poisson case, optimal consumption may occur in gulps and at rates.

4.5.1 Geometric Brownian Motion

For $X = (W(t),\ t \geq 0)$, a Brownian motion, the parameter condition (23) ensuring well-posedness of our problem (2) takes the form

$$\delta > \hat{\delta} = \alpha r + \frac{1}{2}\frac{\alpha\theta^2}{1-\alpha}\,. \tag{44}$$

Note that this is exactly the regularity assumption needed in the context of the classical Merton portfolio problem; compare, e.g., Karatzas and Shreve (1998), Remark 3.9.23, Merton (1990), Section 4.6, or Korn (1997), Corollary 3.3.7.

Recall that the result in Hindy and Huang (1993) is obtained by use of the Bellmann methodology under the additional parameter restriction

$$\delta < r + \beta(1-\alpha)\,; \tag{45}$$

confer their Equation (41). Our approach, however, shows that this assumption can be dispensed with. Only the natural well-posedness condition (44) has to be required; compare Theorem 4.9 and Lemma 4.15.

Let us now focus on the economic interpretation of the results in the Brownian case. Recall that the agent consumes whenever the process

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0\leq s\leq t}\{X(s) + \mu s\}\right)$$

increases. Since X is Brownian motion, μ is given by

$$\mu = \frac{1}{\theta}\left(\tfrac{1}{2}\theta^2 + r + \beta(1-\alpha) - \delta\right)\,.$$

From this we can immediately infer the following fundamental difference between the classic time-additive models and the Hindy-Huang-Kreps (HHK) approach: While, in the time-additive case, agents typically consume all the time, in the HHK-framework, it is typical that optimal consumption occurs periodically (and can even be singular with respect to Lebesgue measure as in the Brownian case considered here). This has already been pointed out by Hindy and Huang (1993). In their case, i.e., when (45) holds true, the agent never refrains from consumption totally. In fact, our analysis shows that this is the case iff $\mu \geq 0$, i.e., iff

$$\delta \leq r + \beta(1-\alpha) + \tfrac{1}{2}\theta^2 .$$

It is interesting to see what happens if this inequality does not hold true. In this case, the overall supremum of the Brownian motion with drift $(W(t) + \mu t,\ t \geq 0)$ is finite almost surely. Therefore, the investor's cumulated total consumption is finite with probability one as well, even though the time horizon is infinite. Moreover, there is an almost surely finite last time of consumption. However, since this is not a stopping time the agent will not consume all his wealth at that time because he does not know for sure that there will not be another opportunity for consumption! To illustrate this point further, let us calculate the optimal portfolio for an agent in a standard Samuelson-type model of the asset market.

Portfolios Consider a complete financial market with one risky asset whose price evolves according to

$$P(0) > 0, \quad dP(t) = P(t)\,(\sigma\, dW(t) + (r + \theta\sigma)\, dt) \quad (t \geq 0)$$

for some $\sigma > 0$. The agent uses the asset and the bond to finance his consumption plan C^L. Under $\mathbb{P}^*$,

$$W^*(t) \stackrel{\Delta}{=} W(t) + \theta t \quad (t \geq 0)$$

becomes a Brownian motion and the discounted asset price $\bar{P} = (e^{-rt}P(t),\ t \geq 0)$ is — as usual — a $\mathbb{P}^*$-martingale with

$$d\bar{P}(t) = \sigma \bar{P}(t)\, dW^*(t) \quad (t \geq 0) .$$

Denote by

$$V^L(t) \stackrel{\Delta}{=} \mathbb{E}^* \left[\int_t^{+\infty} e^{-r(s-t)}\, dC^L(s) \,\middle|\, \mathcal{F}_t \right]$$

the present value of the remaining consumption at time $t \geq 0$. The portfolio strategy π^L in the asset we are looking for has to satisfy

$$dV^L(t) = \pi^L(t)\, d\bar{P}(t) - e^{-rt}\, dC^L(t) \quad (t \geq 0) .$$

Theorem 4.17 *The agent puts a constant fraction of his wealth in the risky asset:*

$$\frac{\pi^L(t)\bar{P}(t)}{V^L(t)} \equiv \frac{\zeta^*}{\sigma} ,$$

where ζ^* is as in (37).

Remark 4.18 *This similiarity to the original Merton portfolio problem has already been obwerved by Hindy and Huang (1993).*

PROOF : We are interested in the representation of the martingale part of V^L as a stochastic integral with respect to W^*. We will therefore compute V^L explicitly.

We have $V^L(0-) = \Psi^L$, which has been computed in (38). For $t > 0$ we proceed along the same lines as in the proof of Theorem 4.9 and the calculation of (38) to obtain:

$$V^L(t) \quad = \quad \frac{e^{-\beta t}}{\beta}\left(\mathbb{E}^*\left[\int_t^{+\infty}(r+\beta)e^{-(r+\beta)(s-t)}A^L(s)\,ds \,\middle|\, \mathcal{F}_t\right] - A^L(t)\right).$$

The above expectation can be rewritten as

$$\mathbb{E}^*\left[A^L(t) \vee L^{-\frac{1}{1-\alpha}} e^{\frac{\theta}{1-\alpha}\left(W^*(t)+\mu^* t+\sup_{0\le s\le\tau^*}\{W^*(t+s)-W^*(t)+\mu^* s\}\right)} \,\middle|\, \mathcal{F}_t\right]$$

where τ^* is an independent exponential random variable with parameter $r+\beta$ and

$$\mu^* \triangleq \frac{r+\beta(1-\alpha)-\delta}{\theta} - \frac{1}{2}\theta\,.$$

The Markov property of Brownian motion and Lemma 4.13 (ii) allow us to conclude that this is equal to

$$A^L(t) + \frac{L^{-\zeta^*/\theta}}{\nu} e^{\zeta^*\mu^* t} A^L(t)^{-\nu} e^{\zeta^* W^*(t)}$$

where ζ^* is determined by (37) and $\nu \triangleq \frac{(1-\alpha)\zeta^*-\theta}{\theta}$, a strictly positive constant because of condition (44). The present value of the consumption policy C^L is therefore given by

$$V^L(t) = \frac{L^{-\zeta^*/\theta}}{\beta\nu} e^{(\zeta^*\mu^*-\beta)t} A^L(t)^{-\nu} e^{\zeta^* W^*(t)}. \tag{46}$$

Hence,

$$dV^L(t) = V^L(t)\zeta^*\,dW^*(t) + \text{terms of bounded variation}$$

and we conclude that at each time $t \ge 0$ the investor must hold

$$\pi^L(t) \triangleq \frac{\zeta^*}{\sigma}\frac{V^L(t)}{\bar{P}(t)}$$

shares of the risky asset in his portfolio in order to finance the consumption policy C^L. □

Remark 4.19 *If $\sigma = \zeta^*$, the agent invests all his wealth in the risky asset. This case can be viewed as a single-agent equilibrium of the stock market for this type of investors.*

Consider again the case when there is an almost surely finite, yet imperceptible last time of consumption. This occurs, as we pointed out above, iff

$$\delta > r + \beta(1 - \alpha) + \tfrac{1}{2}\theta^2.$$

In this case, the investor's level of satisfaction eventually decreases at rate β forever, inducing an ever increasing appetite. His wealth, however, decreases at a higher rate, namely $|\zeta^* \mu^*| + \beta$, as can be read off (46). Thus, the investor's relative level of satisfaction — the fraction of his level of satisfaction and his wealth — remains large. This in turn drives him to wait for better times to come. He keeps being engaged in the risky asset although he knows that with probability one he will be 'unlucky' from some point in time on. This illustrates that, as already noted by Hindy and Huang (1993), an investor whose preferences exhibit local substitution is less risk averse than his time-additive counterpart, because he obtains utility from past consumption which makes him less dependent on current consumption. He can afford to invest in the risky asset and to refrain from consumption for a while in order to speculate on a higher future level of satisfaction.

4.5.2 Geometric Poisson Processes

Let us now study Poisson price processes, i.e., we let $X = (\pm N(t),\ t \geq 0)$. A jump of the process N corresponds to an unpredictable 'price shock' or, in the terminology of Hindy and Huang (1993), an 'information surprise'. We distinguish the two cases where the shocks are 'good' (price decrease) or 'bad' (price increase).

Upward Price Shocks First we consider the case of 'bad' upward price shocks, i.e., $X = (-N(t),\ t \geq 0)$, a Poisson process with downward jumps and intensity $\lambda > 0$ under the objective probability $\mathbb{P}$.

For this choice of X, the optimization problem (2) is well-posed iff

$$\delta > \hat{\delta} = \alpha r + \lambda \left((1-\alpha) e^{-\frac{\alpha\theta}{1-\alpha}} + \alpha e^{\theta} - 1 \right) .$$

As in the Brownian case,

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \mu s\} \right) ,$$

but now

$$\mu \triangleq \frac{1}{\theta} \left(\lambda(e^{\theta} - 1) + r + \beta(1-\alpha) - \delta \right) .$$

In contrast to the Brownian case, it now may happen that $Z = (X(t) + \mu t,\ t \geq 0)$ has decreasing paths only. Indeed, this is the case iff $\mu \leq 0$, i.e., iff

$$\delta \geq \lambda(e^{\theta} - 1) + r + \beta(1-\alpha) .$$

Hence, a very impatient agent (characterized by a high rate of time preference δ), optimally consumes his whole wealth by one single gulp at time $t = 0$. If the agent is not that impatient, then, apart from a possible initial gulp, he only consumes at rates

$$dC^L(t) = \frac{1}{\beta} e^{-\beta t} dA^L(t) = \frac{r + \beta(1-\alpha) - \delta + \lambda\theta}{\beta(1-\alpha)} e^{-\beta t} A^L(t) 1_{\{\dot{A}^L(t) \neq 0\}} \, dt \quad (t > 0)$$

until an upward price shock makes him refrain from consumption. After a while, when his wealth and appetite have become large enough again, he restarts consumption until the next shock, etc.

Downward Price Shocks In the second Poisson example, there are 'nice' downward price shocks, i.e., $X = (N(t),\ t \geq 0)$ with N as before.

As usual,

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \mu s\} \right)$$

where, in this case,

$$\mu \triangleq \frac{1}{\theta} \left(\lambda(e^{-\theta} - 1) + r + \beta(1-\alpha) - \delta \right) .$$

Observe that now X has positive jumps and, therefore, neither Assumption 4.14 (i) nor Assumption 4.14 (ii) holds true. Hence, the closed-form expressions for the prices of optimal consumption plans and their utilities as derived at the end of Section 4.4 are no longer valid here.

However, we still have that the utility maximization problem (2) is well-posed if

$$\delta > \hat{\delta} = \alpha r + \lambda \left((1-\alpha) e^{\frac{\alpha\theta}{1-\alpha}} + \alpha e^{-\theta} - 1 \right) . \tag{47}$$

Remark 4.20 *We conjecture, but cannot yet prove that condition* (47) *is also necessary for well-posedness of problem* (2) *in the case considered here. Note, however, that we know by Theorem 4.9 that the problem is ill-posed if $\delta < \hat{\delta}$. Thus the only open case is $\delta = \hat{\delta}$.*

Economically, it is interesting to note that, depending on the parameter values, two types of (optimal) consumption behavior can emerge in the presence of downward price shocks considered here:

- If we have $\mu \geq 0$, then, once the investor has started consumption, he consumes continually at rates

$$\dot{C}^L(t) = \frac{\lambda(e^{-\theta} - 1) + r + \beta(1-\alpha)}{\beta(1-\alpha)} e^{-\beta t} A^L(t)$$

and takes a gulp

$$\Delta C^L(t) = \frac{e^{\frac{\theta}{1-\alpha}}}{\beta} e^{-\beta t} A^L(t-) \Delta N(t)$$

whenever a nice price shock occurs. This is due to the fact that prices decline very fast and the relative wealth of the consumer increases.

- If the world is not such a comfortable one, i.e., if $\mu < 0$, then the agent consumes only in gulps, namely every time a nice price shock causes A^L to reach a new maximum.

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